

Classical transients and the support of open quantum maps

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(Dated: July 25, 2012)

The basic ingredients in a semiclassical theory are the classical invariant objects serving as a support for the quantization. Recent studies, mainly obtained on quantum maps, have led to the commonly accepted belief that it is the classical repeller – the set of non escaping orbits in the future and past evolution – the object that suitably plays this role in open scattering systems. In this paper we present numerical evidence warning that this may not always be the case. For this purpose we study recently introduced families of tribaker maps [L. Ermann, G.G. Carlo, J.M. Pedrosa, and M. Saraceno, Phys. Rev. E **85**, 066204 (2012)], which share the same asymptotic properties but differ in their short time behavior. We have found that although the eigenvalue distribution of the evolution operator of these maps follows the fractal Weyl law prediction, the theory of short periodic orbits for open maps fails to describe the resonance eigenfunctions of some of them. This is a strong indication that new elements must be included in the semiclassical description of open quantum systems.

PACS numbers: 05.45.Mt, 03.65.Sq

I. INTRODUCTION

The strong recent interest in the study of open quantum systems has evidenced that this area is still in a very early stage of development, specially when compared with the knowledge that we have of the closed counterparts. As a consequence, it has become an extremely active topic in fundamental physics [1]. The most important advance concerns scattering systems, where the fractal Weyl law (FWL) for the number of long-lived resonances has been conjectured and tested in different examples [2]. The corresponding classical invariant distribution, i.e. the fractal hyperbolic set of all the trajectories non-escaping in the past and future (the repeller), plays a fundamental role for the interpretation of the quantum spectrum of quasibound (resonant) states. According to this law [3] the number of long-lived resonances scales as $N^{d/2}$, where d is the fractal dimension of the repeller and N that of the Hilbert space. After the proof of the fractal Weyl upper bound for a Hamiltonian flow showing a fractal trapped set [4], results that support the validity of the conjectured law both for smooth and hard wall potentials [5] have followed. However, open quantum maps provide the ideal arena for this kind of studies, due to its computational simplicity [6], and accordingly we will focus on them in this work.

Another recent advance in the understanding of scattering systems has been the development of the short

periodic orbits (POs) theory for open quantum maps [7]. This theory makes use of the shortest POs living in the repeller to construct a classically motivated basis in which the quantum non unitary operators corresponding to the classical maps can be adequately expressed. It has been shown that there is a connection between the FWL and the possibility to obtain the quantum long-lived resonances by using the detailed classical information of the repeller, i.e. the embedded trajectories [8].

As a result of all this evidence, the repeller is commonly accepted to be the fundamental invariant classical structure in phase space able to explain the quantum mechanics of open systems. However, in a recent publication [9], it has been shown for a set of maps sharing the same repeller and asymptotic features (including the escape rate) but having different classical transient behavior, that the quantum properties of the resonances can differ strongly from this view. This suggests that also transients may play a fundamental role in the description of open quantum systems.

In this paper we further investigate this issue, by presenting a twofold numerical calculation on the tribaker map families introduced in [9]. These families differ in the way in which the openings are defined. In the so called shift family the area of the opening is the same for all of its members, while in the intersection family this area changes. First, we have verified that the resonance distributions follow the FWL for these two kinds of maps. Second, we have tested the ability of the theory of short POs for open quantum maps to reproduce the eigenvalues and the corresponding resonance eigenfunctions. This approach is used as a tool to investigate the

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actual classical support of the quantum resonances, finding that the quantum information of some of the shift family members cannot be reproduced. These results cast doubts on the commonly accepted belief that the repeller is just enough to support the quantization of open maps.

The organization of this paper is as follows: In Sec. II we introduce the features of the shift and intersection families of open tribaker maps. In Sec. III we study the effects of classical transients on the spectral statistics. The same is done in Sec. IV regarding the classical support of open quantum maps by using the theory of short POs as the main tool. Finally, in Sec. V we present a final discussion, and our perspectives for future work.

II. OPEN MAPS: SHIFT AND INTERSECTION FAMILIES

Open maps can be defined as the deterministic evolution corresponding to a given (closed) map followed by the loss of trajectories which pass through a given region in phase-space. The set of trajectories that survive in the future and in the past define the backwards and forward trapped sets, respectively. In turn, the repeller is given by their intersection. The structure of this classical invariant can be very complex for chaotic dynamics, having a fractal dimension in most of the cases. Another relevant property of open maps is the decay rate, which is defined as the asymptotic loss of probability in one iteration.

In this work we will study the shift and intersection families of open tribaker maps defined in [9] that differ only in their short time dynamics, sharing the same repeller and decay rate. They are defined preserving parity symmetry, and therefore with the same openings in position and momentum.

In ternary notation the map action is given by a Bernoulli shift of $q = 0.\epsilon_0\epsilon_1\epsilon_2\dots$ and $p = 0.\epsilon_{-1}\epsilon_{-2}\epsilon_{-3}\dots$ (given by the corresponding trits $\epsilon_j = 0, 1, 2$) as $(p|q) = \dots\epsilon_{-2}\epsilon_{-1}.\epsilon_0\epsilon_1\epsilon_2\dots \rightarrow (p'|q') = \dots\epsilon_{-2}\epsilon_{-1}\epsilon_0.\epsilon_1\epsilon_2\dots$ where the dot is moved one position to the right. Shift and intersection families of open maps can be straightforwardly defined replacing the usual trit ϵ_j by the open trit $\tilde{\epsilon}_j$ where the value 1 is forbidden (i.e., $\tilde{\epsilon}_j = 0, 2$). In this framework, the shift family members (\mathcal{B}_k^s) are defined with two open trits corresponding to the k -th most significant trit of both position and momentum. Equivalently the intersection family members (\mathcal{B}_k^i) have the first k trits open, both in position and momentum. The iterations of open baker maps in the unit square for $k = 1, 2$ and both, shift and intersection families are illustrated in Fig.1.

The quantum version of the shift and intersection map families can be defined from the usual quantum tribaker map in a Hilbert space spanned by l qutrits (with dimen-

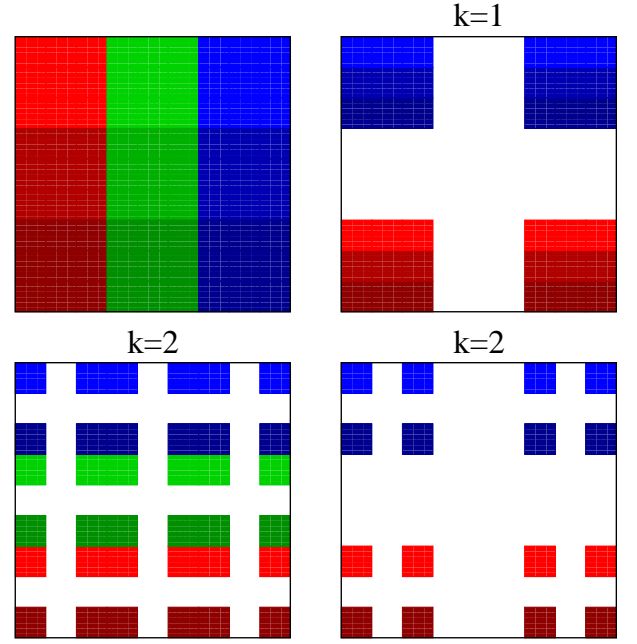


FIG. 1. (color online) Iteration of the classical open baker map. The unit square divided in 9 regions (3 colors with different intensities) is shown in the top-left panel. One iteration of the colored unit square with the open baker map is shown for $k = 1$ (shift or intersection family) in top-right panel and for $k = 2$ for the shift (bottom-left panel) and intersection (bottom-right panel) cases.

sion $N = 3^l$)

$$B_{pos} = G_N^\dagger B_{mix} = G_N^\dagger \begin{pmatrix} G_{N/3} & 0 & 0 \\ 0 & G_{N/3} & 0 \\ 0 & 0 & G_{N/3} \end{pmatrix} \quad (1)$$

where the antiperiodic Fourier transform G_N connects position and momentum eigenvectors ($|q_j\rangle$ and $|p_j\rangle$ with $j = 1, \dots, N$)

$$(G_N)_{j',j} \equiv \langle q_{j'} | p_j \rangle = \frac{1}{\sqrt{N}} e^{-i \frac{2\pi}{N} (j' + \frac{1}{2})(j + \frac{1}{2})}. \quad (2)$$

with $N = 1/(2\pi\hbar)$ (see [10, 11] for quantization details).

The forbidden 1 in one trit $\tilde{\epsilon}$, can be quantized by means of the one qutrit projector $\pi = I - |1\rangle\langle 1| = |0\rangle\langle 0| + |2\rangle\langle 2|$. In this way, the projector applied to the i -th qutrit can be written as

$$\Pi_i = \underbrace{I \otimes \dots \otimes I}_{i-1} \otimes \pi \otimes \underbrace{I \otimes \dots \otimes I}_{l-i} \quad (3)$$

with $\Pi_i = \Pi_i^\dagger$, $\Pi_i^2 = \Pi_i$ and $[\Pi_i, \Pi_j] = 0$ where $i, j = 1, \dots, l$. Following this notation, the shift and intersection families of open quantum tribaker maps can be straightforwardly defined as

$$\tilde{B}_k^s = G_N^\dagger \Pi_k B_{mix} \Pi_k \quad (4)$$

$$\tilde{B}_k^i = G_N^\dagger \Pi_1^\dagger \dots \Pi_{k-1}^\dagger \Pi_k^\dagger B_{mix} \Pi_k \Pi_{k-1} \dots \Pi_1 \quad (5)$$

with $k = 1, \dots, l$, and where the form $\tilde{B}_{mix} = \Pi B_{mix} \Pi$ preserves the parity symmetry of the closed map.

III. TRANSIENT EFFECTS AND THE STATISTICS OF EIGENVALUES OF OPEN QUANTUM MAPS

The geometry of phase space imposes restrictions in the asymptotic distributions of the eigenstates of quantum systems. In bounded systems of f degrees of freedom, the well known Weyl law establishes that the number of eigenstates $N(E)$ of energy less than E is given by the number of cells of size h^f contained in the corresponding allowed volume of phase space $V(E)$, i.e.

$$N(E) = V(E)/(2\pi\hbar)^f. \quad (6)$$

In open systems the relation between the geometry of the phase space and the statistics of eigenvalues is more involved. In these systems, the quantum evolution is given by a nonunitary operator with right and left decaying nonorthogonal eigenfunctions. The corresponding eigenvalues z_n are complex numbers that fall inside the unit circle, that is, $|z_n|^2 \leq 1$. The resonances are conjectured to be localized on the trapped sets [12–14]. This structure of the phase space has led to the prediction of a relation between the number of long-lived resonances and the fractal dimension d of the repeller. As previously mentioned, the FWL establishes that the number of resonances with decay factor $\mu_n = |z_n| = \exp[-\Gamma_n/2]$ scales as $N_\mu \sim N^{d/2}$. This conjecture has been investigated in many systems, including quantum maps, 2-D and 3-D billiards, and the modified Henon-Heiles potential [3, 5, 6, 15]. Also, more scarring has been predicted in open systems than in closed ones [16, 17]; this phenomenon has deep consequences in many applications, such as microlasers [18].

The goal of this Section is to test the validity of the FWL in the families of open tribaker maps introduced earlier. We remark that all these systems have the same repeller, classical information that determines the statistical behavior of the resonances which is embodied in the FWL. In Fig. 2 we show the ordered decay factor μ_n for the usual open tribaker map (i.e., the first member of any of the two families, $\tilde{B}_1^{s/i}$), having Hilbert space dimensions $N = 3^l$ with $l = 4, 5, 6, 7, 8$ and 9 . In the inset of Fig. 2 we show the number of resonances with $\Gamma_n > 0.1$ as a function of N . It is clear that $N_\mu \sim N^{d/2}$ with $d = \ln(2)/\ln(3)$ the fractal dimension of the repeller as predicted by the FWL.

In Fig. 3 we show the same statistics, but for the eigenvalues of the open map \tilde{B}_4^s , i.e. the 4th member of the shift family. It is worth mentioning that we have also studied the other members of this family obtaining similar qualitative results as those shown in Fig. 3. We clearly see that the ordered decay factor for this map using different Hilbert space dimensions roughly behaves

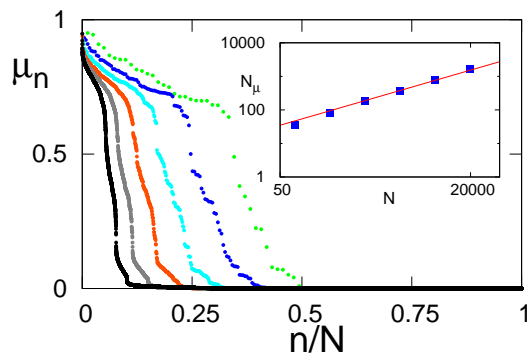


FIG. 2. (color online) Ordered decay factor μ_n of the usual open tribaker map as a function of n/N . Hilbert dimensions $N = 3^l$ with $l = 4, 5, 6, 7, 8$ and 9 (green, blue, turquoise, red, gray, and black symbols, respectively). Inset: Fraction of states N_μ with decay rate $\Gamma > 0.1$ as a function of the Hilbert space dimension. The solid red line corresponds to the prediction of the FWL ($N_\mu \sim N^{\ln(2)/(2 \ln(3))}$).

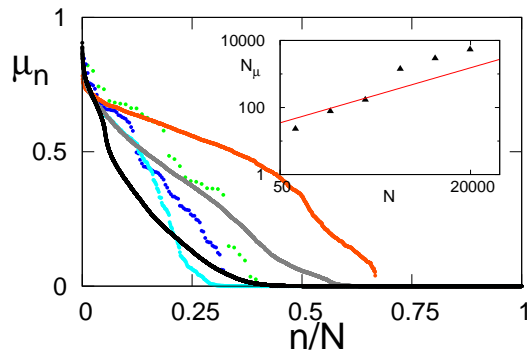


FIG. 3. (color online) Ordered decay factor μ_n of the open map \tilde{B}_4^s as a function of n/N . The Hilbert space dimension is $N = 3^l$ with $l = 4, 5, 6, 7, 8$ and 9 (green, blue, turquoise, red, gray, and black symbols, respectively). Inset: Fraction of states N_μ with decay rate $\Gamma > 0.1$ as a function of Hilbert space dimension. The solid red line corresponds to the prediction of the FWL ($N_\mu \sim N^{\ln(2)/(2 \ln(3))}$).

as in the case of the usual open tribaker map, i.e. it follows the FWL.

Finally, we consider the statistics of the eigenvalues in the case of the intersection family. In Fig. 4 the number of resonances with $\Gamma > 0.1$ as a function of N for the open maps \tilde{B}_k^i with $k = 1, 2, 3, 4$ is shown. We can see that again the number of resonances as a function of N seems to follow the prediction of the FWL.

In the following Section we go deeper into the study of the classical support of open quantum maps.

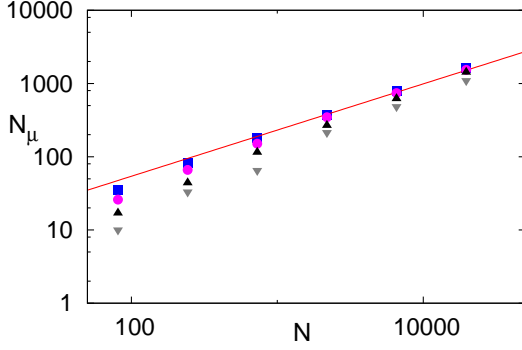


FIG. 4. (color online) Fraction of states N_μ with decay rate $\Gamma > 0.1$ as a function of the Hilbert space dimension for the open maps \tilde{B}_k^i with $k = 1, 2, 3, 4$ (boxes, circles, up and down triangles, respectively).

IV. TRANSIENT EFFECTS AND THE THEORY OF SHORT PERIODIC ORBITS FOR OPEN QUANTUM MAPS

The recently developed theory of short POs for open quantum maps [7, 8] (which finds its roots in the corresponding theory for closed systems [19]) provides the ideal tool to test up to which point the repeller, more specifically the POs living in it are the actual classical support of quantum resonances. The essential ingredient of this approach is the open scar function associated to each one of these trajectories.

Taking γ as a periodic orbit of fundamental period L that belongs to an open map, we define coherent states $|q_j, p_j\rangle$ associated to each point of the orbit (it has a total of L points, all in the repeller) and construct the linear combination

$$|\phi_\gamma^m\rangle = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} \exp\{-2\pi i(jA_\gamma^m - N\theta_j)\} |q_j, p_j\rangle. \quad (7)$$

In this expression $m \in \{0, \dots, L-1\}$ and $\theta_j = \sum_{l=0}^j S_l$, where S_l is the action acquired by the l th coherent state in one step of the map. The total action is $\theta_L \equiv S_\gamma$ and $A_\gamma^m = (NS_\gamma + m)/L$. The right and left scar functions for the periodic orbit are defined through the propagation of these linear combinations under the open map \tilde{U} (up to approximately the system's Ehrenfest time τ).

$$|\psi_{\gamma,m}^R\rangle = \frac{1}{\mathcal{N}_\gamma^R} \sum_{t=0}^{\tau} \tilde{U}^t e^{-2\pi i A_\gamma^m t} \cos\left(\frac{\pi t}{2\tau}\right) |\phi_\gamma^m\rangle, \quad (8)$$

and

$$\langle\psi_{\gamma,m}^L| = \frac{1}{\mathcal{N}_\gamma^L} \sum_{t=0}^{\tau} \langle\phi_\gamma^m| \tilde{U}^t e^{-2\pi i A_\gamma^m t} \cos\left(\frac{\pi t}{2\tau}\right). \quad (9)$$

Normalization ($\mathcal{N}_\gamma^{R,L}$) is chosen in such a way that $\langle\psi_{\gamma,m}^R|\psi_{\gamma,m}^R\rangle = \langle\psi_{\gamma,m}^L|\psi_{\gamma,m}^L\rangle$ and $\langle\psi_{\gamma,m}^L|\psi_{\gamma,m}^R\rangle = 1$. We

select a number of short POs that approximately covers the repeller, whose number scales following the FWL prediction [8]. Then, we construct an appropriate basis in which we can write open evolution operators associated to open maps. After solving a generalized eigenvalue problem we obtain a classically motivated approximation to the system's long-lived resonances [7].

The short periodic orbit theory is strongly relying on the repeller as the source of fundamental information in order to describe the long-lived portion of the spectra of open maps. In fact, it takes into account the details of the open dynamics on the repeller, going beyond the consideration of just a measure such as, for instance, the fractal dimension which governs the FWL. But now the question arises: Is this enough?, or are more elements needed?

We define the performance P of this method as the fraction of long-lived eigenvalues that it is able to reproduce up to an error given by $\epsilon = \sqrt{(\text{Re}(z_i^{ex}) - \text{Re}(z_i^{PO}))^2 + (\text{Im}(z_i^{ex}) - \text{Im}(z_i^{PO}))^2}$, where z_i^{ex} and z_i^{PO} are the exact eigenvalues and those given by the short POs theory, respectively. Hereafter, we take $\epsilon = 0.001$ and consider just the eigenvalues with modulus greater than 0.01. These threshold values guarantee good performance of the approximation and the evaluation of a meaningful number of eigenvalues in the whole range of maps that we have studied. However, if detailed comparisons are needed, these values should be carefully adapted to each map, this going beyond purposes of the present paper. In Figs. 5 and 6 we show P as a function of the number of used POs, N_{POs} ; here our method is applied to the shift and intersection families for $N = 3^5$, respectively. At this relative low value of N it is already evident that the usual open tribaker map that corresponds to the $k = 1$ value for both families proves to be well reproduced by this approximation. In the case of the shift family the good performance is suddenly lost when we arrive at the $k = l/2$ threshold, i.e., from $k = 3$ on, the method clearly fails. We notice that the resonances of the shift family occupy classically forbidden regions for $k \geq l/2$ [9]. Nevertheless, the performance is good for all the members of the intersection family. We conjecture that this is because the probability outside the region roughly corresponding to the classical repeller is erased by means of the projector. To further confirm this conjecture we have made the same calculation for the $l = 6$ case, which qualitatively coincides with our previous reasoning. The corresponding results can be seen in Figs. 7 and 8.

In order to provide further insight on the differences among members of these two families of maps and relate them to basic features of general open (scattering) quantum systems we will study the phase space distributions associated to the right and left eigenfunctions. For this purpose we use a recently introduced representation [20] that reveals the part of the quantum probability which is in the closest correspondence to the classical repeller. By defining the symmetrical operator \hat{h}_j associated to

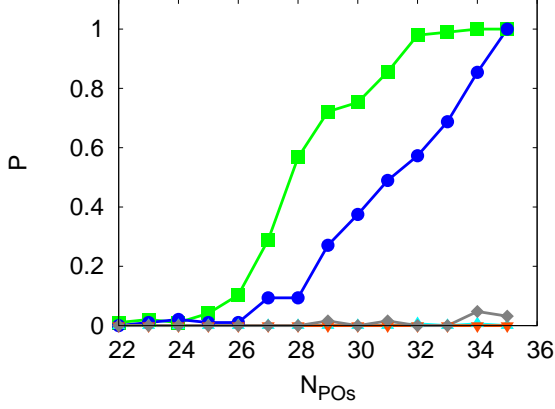


FIG. 5. (color online) Performance P of the short POs approach for the case of the shift family: fraction of long-lived resonances found as a function of the number of POs used in the calculations N_{POs} . Hilbert space dimension is $N = 3^l$ with $l = 5$, all possible k are shown. (Color) gray lines with squares correspond to $k = 1$, circles to $k = 2$, triangles up to $k = 3$, triangles down to $k = 4$, and diamonds to $k = 5$.

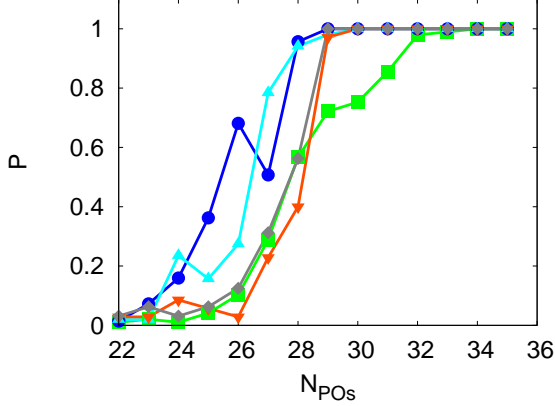


FIG. 6. (color online) Performance P of the short POs approach for the case of the intersection family: fraction of long-lived resonances found as a function of the number of POs used in the calculations N_{POs} . Hilbert space dimension is $N = 3^l$ with $l = 5$, all possible k are shown. We use the same colors and symbols as in Fig. 5.

the right $|R_j\rangle$ and left $\langle L_j|$ eigenstates

$$\hat{h}_j = \frac{|R_j\rangle\langle L_j|}{\langle L_j|R_j\rangle} \quad (10)$$

which is related to the eigenvalue λ_j , we construct the sum of the first j of these projectors, ordered by decreasing modulus of the corresponding eigenvalues ($|\lambda_j| \geq |\lambda_{j'}|$ with $j \leq j'$)

$$\hat{Q}_j \equiv \sum_{j'=1}^j \hat{h}_{j'}. \quad (11)$$

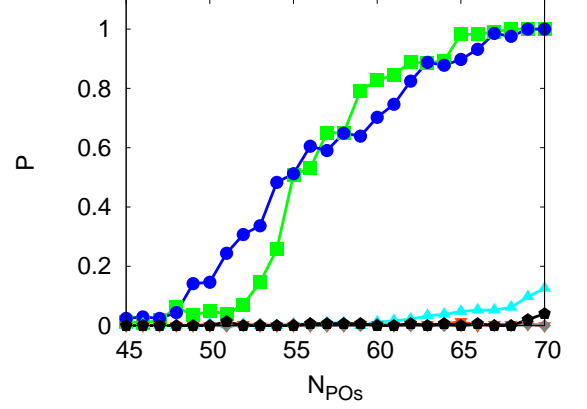


FIG. 7. (color online) Performance P of the short POs approach in the case of the shift family: fraction of long-lived resonances found as a function of the number of POs used in the calculations N_{POs} . Hilbert space dimension is $N = 3^l$ with $l = 6$, all possible k are shown. (Color) gray lines with squares correspond to $k = 1$, circles to $k = 2$, triangles up to $k = 3$, triangles down to $k = 4$, diamonds to $k = 5$, and pentagons to $k = 6$.

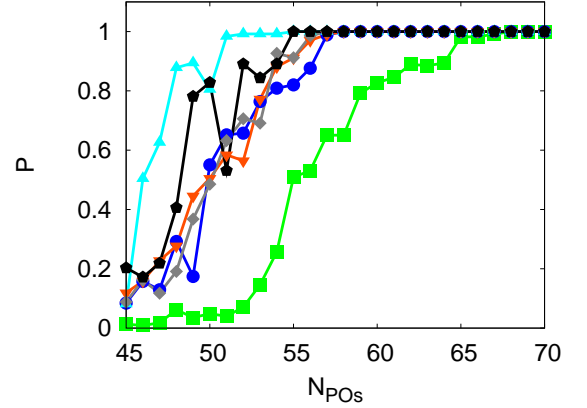


FIG. 8. (color online) Performance P of the short POs approach in the case of the intersection family: fraction of long-lived resonances found as a function of the number of POs used in the calculations N_{POs} . Hilbert space dimension is $N = 3^l$ with $l = 6$, all possible k are shown. We use the same colors and symbols as in Fig. 7.

The object to study consists of their phase space representation by means of coherent states $|q, p\rangle$, which is given by

$$h_j(q, p) = |\langle q, p | \hat{h}_j | q, p \rangle| \quad (12)$$

$$Q_j(q, p) = |\langle q, p | \hat{Q}_j | q, p \rangle|. \quad (13)$$

In Fig. 9 we show Q_{32} for the exact resonances and the ones given by the short POs approach, obtained for $l = 5$ and $N_{POs} = 32$ (it is worth mentioning that we have observed similar results for Q_j with j around 32). In the

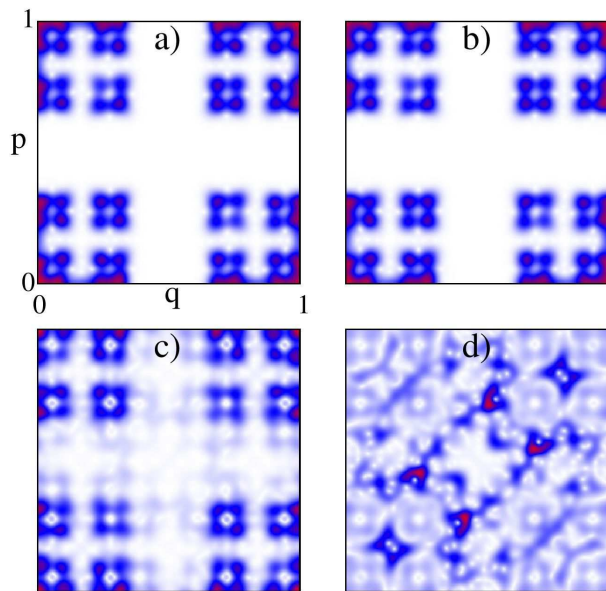


FIG. 9. (color online) Phase space representation of $Q_{32} = \sum_{j=1}^{32} h_j$ for the exact and short POs approach with $l = 5$ and $N_{POs} = 32$. The map with $k = 1$ (both shift and intersection families) is shown in panels a) for exact eigenstates and b) for the short POs approach. For the case of shift family map with $k = 3$ panel c) shows the exact eigenstates and d) the short POs approach. The intensities of phase space representation are white for zero value and red (dark-gray) for maximum.

upper panels the case $k = 1$ (that coincides for both the shift and intersection families) makes clear that the short POs theory is very accurate in reproducing not only the long-lived sector of the spectrum but also the component of the resonance eigenfunctions that live on the classical repeller. In fact, panels a) and b) are almost indistinguishable. This shows undoubtedly that the classical repeller is the meaningful support of long-lived states in this case. Nevertheless, although we do not show here results for the intersection family, they all reproduce very well these components of the eigenstates, in agreement with the conclusions previously shown by means of the spectra. When the shift family is analyzed, again a sort

of phase transition behavior is observed, that spoils the short POs theory results in the same fashion as for the spectral values. This happens at $k \geq l/2$; here $l/2 = 3$, case which we show in Figs. 9 c) and d).

V. CONCLUSIONS

At present, it is commonly accepted that the classical information contained in the repeller is the fundamental ingredient to describe quantum chaotic scattering systems. However, a recent study [9] has shown that transient features of the classical dynamics have crucial effects in the resonances of the quantum system.

In this paper we put this conclusion on firmer grounds. First, we have tested the validity of the FWL, one of the cornerstones of open chaotic quantum systems theory in a family of quantum open tribaker maps [9]. We have clearly shown that the statistics of the eigenvalues follows the FWL behavior. In order to unveil the classical support of the resonances we have attempted to obtain them by means of the short POs approach. This theory has been recently linked to the FWL in Ref. [8], and it is based on the short POs that live in the repeller. We have defined a measure for its performance, showing that it fails to describe the quantum resonances associated to $k \geq l/2$ for the shift family of maps. In contrast, the intersection family, whose opening forces the probability to remain roughly inside what can be called the classical repeller's area, is very well reproduced.

Summarizing, our results in open quantum maps have shown the need to include other ingredients, either classical and/or quantum, additionally to the repeller, in order to satisfactorily describe the quantum mechanics of scattering systems. Further investigations are needed in order to find them and extend the short POs theory for open quantum maps.

VI. ACKNOWLEDGMENTS

This work was supported by the MICINN-Spain under contract MTM2009-14621, CEAL, and also CONICET, UBACyT and ANPCyT (from Argentina).

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